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THE NUMBER OF EDGES OF MANY FACES IN A LINE SEGMENT ARRANGEMENT[1]

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We show that the maximum number of edges bounding m faces in an arrangement of n line segments in the plane is $O(m^{2/3}n^{2/3}+n\alpha(n)+n\log m)$. This improves a previous upper bound of Edelsbrunner et al. [5] and almost matches the best known lower bound which is $\Omega(m^{2/3}n^{2/3}+n\alpha(n))$. In addition, we show that the number of edges bounding any m faces in an arrangement of n line segments with a total of t intersecting pairs is $O(m^{2/3}t^{1/3}+n\alpha(\frac{t}{n})+n\min\{\log m,\log\frac{t}{n}\})$, almost matching the lower bound of $\Omega(m^{2/3}t^{1/3}+n\alpha(\frac{t}{n}))$ demonstrated in this paper.

1. Introduction

Let S be a finite set of line segments in the Euclidean plane, where a *line segment* is a connected, closed, bounded subset of a line. We call the subdivision of the plane induced by these line segments the *arrangement* $\mathcal{A}(S)$ of S; it consists of vertices, edges, and faces. The total number of edges bounding some collection of faces in this arrangement is called the *combinatorial complexity* of the collection. We write $K(m,n)$ for the maximum combinatorial complexity of any m faces in any arrangement of n line segments. Note that $\varkappa(n)=\binom{n}{2}-n+2$ is the maximum possible number of faces in an arrangement of n line segments. For $m=\varkappa(n)$ we clearly have $K(m,n)=\Theta(n^2)$, while for a single face, $K(1,n)=\Theta(n\alpha(n))$, where $\alpha(n)$ denotes the functional inverse of Ackermann's function (see [11], [9], [12]).[2]

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[2]For technical reasons that will become more apparent later, we modify the standard definition of $\alpha(\cdot)$ in the following manner which does not affect its asymptotic behavior. Consider Ackermann's function and extend its domain to include all non-negative real numbers by a linear interpolation

A recent paper [5] shows that $K(m,n) = O(m^{2/3-\delta}n^{2/3+2\delta} + n\alpha(n)\log m)$,[3] for any $\delta > 0$, where the constant of proportionality depends on δ. For most values of m this bound exceeds the best known lower bound which is $\Omega(m^{2/3}n^{2/3} + n\alpha(n))$. Another recent paper [3] demonstrates a tight $\Theta(m^{2/3}n^{2/3}+n)$ bound for the maximum combinatorial complexity of m faces in an arrangement of n lines.

In this paper we show that $K(m,n) = O(m^{2/3}n^{2/3}+n\alpha(n)+n\log m)$. This bound improves that of [5] and differs from the lower bound mentioned above only in the term $n\log m$ which dominates the other two terms only if $2^{\alpha(n)} \leq m \leq n^{1/2}\log^{3/2} n$. For instance, it yields $K(1,n) = O(n\alpha(n))$ and $K(\varkappa(n),n) = O(n^2)$, in agreement with the known and tight results mentioned above.

We also consider the case of arrangements of line segments with fewer than quadratically many intersecting pairs. Using a mixture of our techniques and those of [3], we show that the maximum combinatorial complexity of m faces in an arrangement of n line segments with $t \leq \binom{n}{2}$ intersecting pairs is $O(m^{2/3}t^{1/3}+n\alpha(\frac{t}{n})+ n\min\{\log m, \log\frac{t}{n}\})$ and $\Omega(m^{2/3}t^{1/3}+n\alpha(\frac{t}{n}))$. This result has applications to bounding the combinatorial complexity of non-convex cells in arrangements of triangles in three-dimensional space (see [1] for details).

The main technical difference between the case of lines and of line segments is that in the former case a fairly early result by Canham [2] shows that $K(m,n) = O(mn^{1/2} + n)$, which is $O(n)$ when $m = O(n^{1/2})$. This property is crucial for the analysis in [3]. The lack of a similar bound (called a "Canham threshold" in [3]) for line segments has so far prevented the direct application of the techniques of [3] to the case of line segments. In lieu, [5] developed a different technique in order to obtain the bound stated above. The results of this paper narrow the gap between the upper and lower bound on the maximum combinatorial complexity of m faces in an arrangement of n line segments and are tight unless $2^{\alpha(n)} \leq m \leq n^{1/2}\log^{3/2} n$.

The paper is organized as follows. Section 2 presents a "Canham threshold" for line segment arrangements that is almost tight and the proof of the first bound mentioned in the abstract. Section 3 extends this bound to the case of arrangements of line segments with few intersecting pairs, and Section 4 concludes with a discussion of the results and some open problems.

2. General Collections of Line Segments

In this section we prove the main result of this paper, the upper bound on $K(m,n)$ mentioned in the abstract and the introduction. We begin the proof by recalling the following lemma of [5], called "combination lemma", restated in a manner suitable for our subsequent analysis.

Lemma 2.1. *(Combination lemma). Let B and R be two collections of $|B|$ blue and $|R|$ red line segments in the plane, $n = |B|+|R|$, and let P be a set of m points in the*

between consecutive integers. Define $\alpha(\cdot)$ to be 1 larger than the functional inverse of this extended Ackermann's function. Thus, $\alpha(\cdot)$ is a continuous and piecewise linear function defined on the set of non-negative real numbers and bounded away from zero.

[3] In this paper, all logarithms are to the base 2.

plane not lying on any of the line segments. Let K_B (K_R) denote the combinatorial complexity of the faces of the arrangement $\mathcal{A}(B)$ ($\mathcal{A}(R)$) that contain points of P, and suppose that the number of faces of the overlay arrangement, $\mathcal{A}(B \cup R)$, that contain points of P is $\overline{m} \le m$. Then the combinatorial complexity of these faces is at most $K_B + K_R + O(\overline{m} + n)$.

The proof is omitted because it is identical to the proof given in [5], with the additional observation that the analysis given there depends only on the number $\overline{m}$ of faces in the overlay arrangement that contain the points of P, and not on the size of P.

In addition, we will need the following fact concerning collections L of lines in *general position*, that is, such that no two lines in L are parallel and no three meet in a common point. For reasons that will become clear later we use the term *funnels* for the faces of the line arrangement $\mathcal{A}(L)$. For a collection F of funnels of $\mathcal{A}(L)$ we say that a line $\ell \in L$ *splits* F if F contains at least one funnel on each side of ℓ. Let T be a binary tree whose leaves are in one-to-one correspondence with the funnels of $\mathcal{A}(L)$. We write $F_\lambda = \{f\}$ if the funnel f corresponds to the leaf λ of T and we define $F_\varkappa = F_\mu \cup F_\nu$ for each interior node $\varkappa$ with children μ and ν. We are now ready to state the fact.

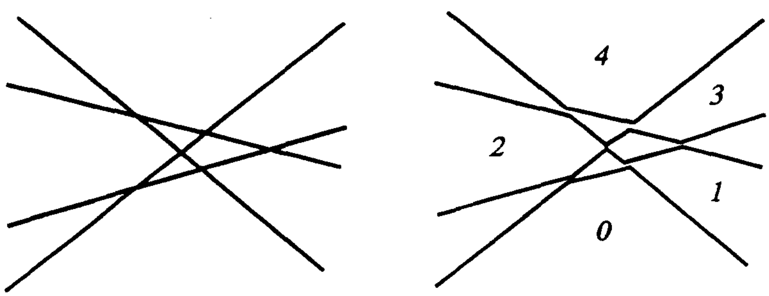

Fig. 2.1. The illustration to the right emphasizes the belts of the line arrangement shown to the left.

Lemma 2.2. *Let L be a set of m lines in general position in the plane. There exists a binary tree T representing the funnels of $\mathcal{A}(L)$ as described above so that*

(a) *the height of T is $O(\log m)$, and*

(b) *for each node $\varkappa$, the collection of funnels $F_\varkappa$ is split by at most $2|F_\varkappa| - 2$ lines in L.*

Proof. Without loss of generality we can assume that no line in L is vertical. We begin by defining the notion of the belt-number of a funnel f in $\mathcal{A} = \mathcal{A}(L)$. Consider a half-line emanating vertically downward from a point in the interior of f. The *belt-number* of f is the number of lines in L that intersect this half-line; it is independent of the choice of the interior point. For $i = 0, 1, \ldots, m$, the *i-belt* of $\mathcal{A}$ is the collection of funnels of $\mathcal{A}$ with belt-number i (refer to Figure 2.1).

To produce the required tree T, we proceed as follows. First, for each i, construct a minimum-height binary tree $T(i)$ whose leaves correspond to the funnels of the i-belt of $\mathcal{A}$, so that the inorder traversal of $T(i)$ visits the funnels from left to right. T is then constructed with the trees $T(i)$ as leaves in a similar manner, using the top-to-bottom order of the belts (see Figure 2.2). The height of T is at most $3\lceil \log m \rceil$ which shows (a). Moreover, T satisfies (b) which can be seen as follows. For a node $\varkappa$ in T with two children μ and ν, the collections F_μ and F_ν represent either two consecutive portions of one belt, or two adjacent groups of consecutive belts. In the former case, the only lines of L that possibly split $F_\varkappa$ but not F_μ or F_ν are the two lines that intersect in the vertex shared by the two portions of the belt. Hence, the number of lines splitting $F_\varkappa$ is at most 2 plus the number of lines that split F_μ plus the number of lines that split F_ν. If F_μ and F_ν represent adjacent groups of belts, then there is no line that separates them, which implies that there is no new line splitting $F_\varkappa$. If $\varkappa$ has only one child μ then $F_\varkappa = F_\mu$, and if $\varkappa$ is a leaf then no line splits $F_\varkappa$. Claim (b) follows by induction on the size of the sets F. ∎

We now present the key lemma (the "Canham threshold") required for the proof of the main result of this section. It makes crucial use of duality and the above two lemmas.

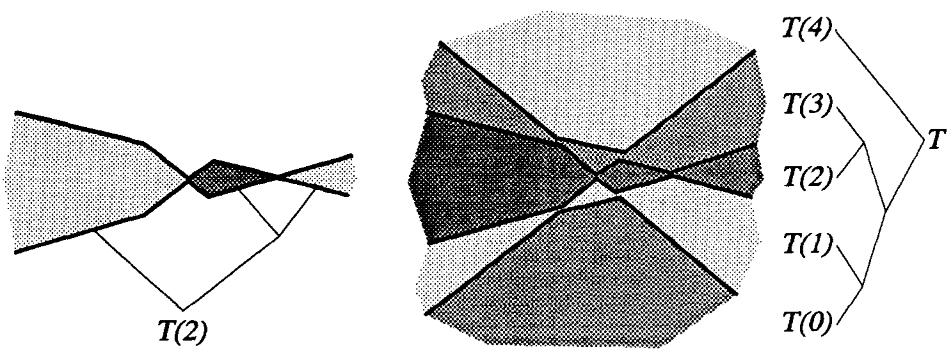

Fig. 2.2. The left shows a tree for a belt, and the right shows how the trees for the individual belts are collected to form a single tree.

Lemma 2.3. *$K(m,n) = O(n \log m + n\alpha(n))$ for $m \le \sqrt{n}$.*

Proof. Let S denote the given collection of n line segments. Mark the desired faces by choosing an interior point in each, thus forming a set P of m points. We have plenty of freedom in choosing the exact locations of the points and can thus assume that no two points lie on a common vertical line and no three points are collinear. We can also assume that no point lies on a line spanned by one of the line segments. Next, we pass to the dual plane. We do this by mapping every point $p = (a,b)$ to the *dual line* $p^* = \{(x,y) | y = ax - b\}$. Note that by definition of the map none of the dual lines is vertical; we can thus unambiguously talk about points that lie (vertically) above or below a dual line. Similarly, a non-vertical line ℓ in the primal plane is mapped to the *dual point* ℓ^* in the dual plane so that $(\ell^*)^* = \ell$. Observe that the duality preserves incidences and 'sidedness', that is, a point p lies on (above, below) a non-vertical line ℓ if and only if the dual point ℓ^* lies on (respectively, above, below) the dual line p^*.

The set of lines $P^* = \{p^* | p \in P\}$ partitions the dual plane into τ funnels $f_1, f_2, \ldots, f_\tau$. We have $\tau = \binom{m}{2} + m + 1$ because the m lines in P^* are in general position which is a consequence of the assumptions on the points in P. Let $S_\lambda \subseteq S$ be the set of line segments so that the points dual to the lines containing them lie in f_λ, for $1 \le \lambda \le \tau$, and set $n_\lambda = |S_\lambda|$. By assumption, no point dual to such a line lies on any line in P^* which implies $\sum_{\lambda=1}^{\tau} n_\lambda = n$.

Now apply the combination lemma (Lemma 2.1) to the collections S_λ, by combining them in pairs, according to the structure of the binary tree T described in Lemma 2.2. This way we obtain new collections of line segments which are then combined in pairs, and so on until all subarrangements are merged to yield the original line segment arrangement. The leaves of T correspond to the sets S_λ, and its root ρ corresponds to the entire set S of line segments. For each node $\varkappa$ with children μ and ν define $S_\varkappa = S_\mu \cup S_\nu$, set $n_\varkappa = |S_\varkappa|$, and let $\overline{m}_\varkappa$ be the number of faces of $\mathcal{A}(S_\varkappa)$ that contain points of P. Let $K_\varkappa$ be the combinatorial complexity of these $\overline{m}_\varkappa$ faces. By the combination lemma we have

$$K_\varkappa = K_\mu + K_\nu + O(\overline{m}_\varkappa + n_\varkappa).$$

Repeated application of this bound yields

$$K_\rho = \sum_{\lambda \text{ a leaf}} K_\lambda + O\left(\sum_{\varkappa \text{ a non-leaf}} \overline{m}_\varkappa\right) + O(n \log m), \tag{1}$$

where the last term bounds the sum of the $n_\varkappa$ and is due to the fact that the height of T is $O(\log m)$ and each line segment belongs to the nodes along one path of T only.

Next we estimate the sum of the $\overline{m}_\varkappa$. Recall that $F_\varkappa$ is the set of funnels that correspond to the leaves in the subtree with root $\varkappa$. By Lemma 2.2(b) at most $2|F_\varkappa| - 2$ lines in P^* split $F_\varkappa$. Each remaining line of P^* either passes above all funnels in $F_\varkappa$ or it passes below all such funnels. In any case, its corresponding point in P lies in the unbounded face of $\mathcal{A}(S_\varkappa)$. Hence, $\overline{m}_\varkappa \le (2|F_\varkappa| - 2) + 1 < 2|F_\varkappa|$. Now we sum this bound over all nodes of T. Because T has only $O(m^2)$ nodes and its height is only $O(\log m)$ we obtain

$$\sum_\varkappa \overline{m}_\varkappa < \sum_\varkappa 2|F_\varkappa| = O(m^2 \log m).$$

All we still need to derive the assertion from (1) is a bound on the sum of the K_λ, over all leaves λ. Each λ corresponds to a funnel f_λ which is a face of $\mathcal{A}(P^*)$. Hence, each point of P lies in the unbounded face of $\mathcal{A}(s_\lambda)$. This implies $K_\lambda = O(n_\lambda \alpha(n_\lambda))$. The sum of the K_λ, over all leaves λ, is $O(n\alpha(n))$ because $\sum_{\lambda=1}^{\tau} n_\lambda = n$ as noted earlier. We can now substitute all bounds into (1) and obtain

$$K_\rho = O(n\alpha(n) + m^2 \log m + n \log m).$$

For $m \le \sqrt{n}$ this implies

$$K(m,n) = O(n \log m + n\alpha(n)),$$

as asserted. ∎

If $m > \sqrt{n}$ we can partition the set of faces into about $\frac{m}{\sqrt{n}}$ groups of at most $\sqrt{n}$ faces each. If we apply Lemma 2.3 to each group we get

$$K(m,n) \leq \left\lceil \frac{m}{\sqrt{n}} \right\rceil K\left(\lfloor\sqrt{n}\rfloor, n\right) = O\left(m\sqrt{n}\log n + m\sqrt{n}\alpha(n)\right) = O\left(m\sqrt{n}\log n\right).$$

Since for $\sqrt{n} < m \leq \varkappa(n)$ we have $\log n = O(\log m)$, we obtain the following bound which is more general than Lemma 2.3.

Corollary 2.4. $K(m,n) = O(m\sqrt{n}\log m + n\log m + n\alpha(n))$.

We are now ready for the main result of this section. In contrast to Lemma 2.3 its proof can conveniently be stated without reference to duality. Indeed, it follows the general paradigm developed in [3]. Thus, the main steps of the proof are to partition the problem into fairly independent subproblems, to use Canham threshold (Corollary 2.4) for each subproblem, and to use bounds on so-called zones to handle the (limited) interaction between the subproblems. The partition will be defined in terms of a *trapezoidation* of the plane, that is, a collections of open and pairwise disjoint trapezoids whose closures cover the entire plane. Here a *trapezoid* is defined as the intersection of four open half-planes, one to the left of a vertical line, one to the right of a vertical line, one below a non-vertical line, and one above a non-vertical line; some of these half-planes can degenerate to the entire plane. In reference to their function the trapezoids will also be called funnels.

Theorem 2.5. $K(m,n) = O(m^{2/3}n^{2/3} + n\alpha(n) + n\log m)$.

Proof. The case $m \leq \sqrt{n}$ follows trivially from Lemma 2.3, so in the remainder of the proof we assume that $m > \sqrt{n}$.

Suppose the m desired faces of the line segment arrangement $\mathcal{A} = \mathcal{A}(S)$ are marked by a collection P of m points each one lying in the interior of the face it specifies. Let r be a fixed integer between 1 and n that will be specified later. By a result of Matoušek [10] there is a constant c and a trapezoidation of the plane with the following properties.

(i) The number of trapezoids (funnels) is $O(r^2)$.

(ii) Each funnel meets at most $\frac{cn}{r}$ of the line segments in S.

(In fact, [10] proves the existence of a trapezoidation so that (ii) even holds for the lines that contain the line segments in S.) If a funnel contains more than $\frac{m}{r^2}$ points of P or more than $\frac{2n}{r^2}$ endpoints of line segments in S we further subdivide it by a vertical cut so that the number of points (respectively, segment endpoints) in each of the resulting two funnels is at most half of what it was for the original funnel. After repeating this operation we finally arrive at a refined trapezoidation, $\mathcal{T}$, with τ funnels $f_1, f_2, \ldots, f_\tau$. Define $m_i = |P \cap f_i|$, let n_i be the number of line segments in S that intersect but do not have an endpoint in f_i, and let k_i be the number of line segments in S that have at least one endpoint in f_i. Then $\mathcal{T}$ satisfies the following properties.

(i) $\tau = O(r^2)$,
(ii) $n_i \leq \frac{cn}{r}$ for each $1 \leq i \leq \tau$, and
(iii) $m_i \leq \frac{m}{r^2}$ and $k_i \leq \frac{2n}{r^2}$ for each $1 \leq i \leq \tau$.

Furthermore, $\sum_{i=1}^{\tau} k_i \leq 2n$ and we can assume that $\sum_{i=1}^{\tau} m_i = m$ because we are free to perturb the points of P within a sufficiently small neighborhood of their initial positions.

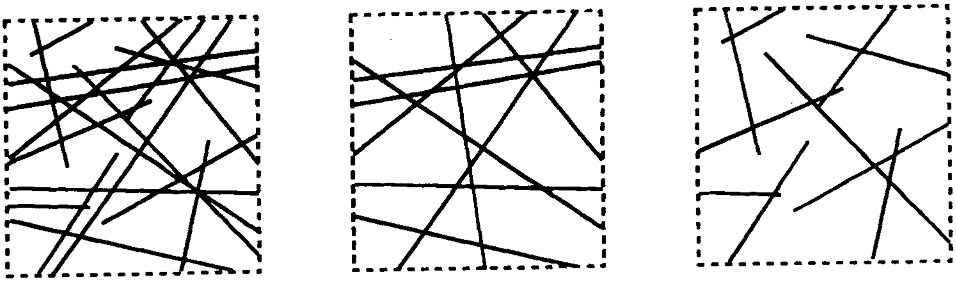

Fig. 2.3. The funnel f_i with its partition $\mathcal{A}_i$ defined by the intersecting line segments is shown to the left. The coarser partitions $\mathcal{A}_i'$ and $\mathcal{A}_i''$ of f_i are shown in the middle and to the right.

As mentioned earlier, $\mathcal{F}$ is used to decompose the global problem into smaller subproblems, one for each funnel. As we will see later these subproblems are not completely independent, but for now let us ignore this issue and consider the subproblem defined by a typical funnel f_i. Let $\mathcal{A}_i$ be the subdivision of f_i defined by the $n_i + k_i$ line segments in S that intersect f_i. We bound the combinatorial complexity of the marked faces of $\mathcal{A}_i$ by separately considering the partitions $\mathcal{A}_i'$ of f_i defined by the n_i line segments that meet f_i but have no endpoint in f_i and $\mathcal{A}_i''$ of f_i defined by the k_i line segments that each have at least one endpoint in f_i (see Figure 2.3). There are m_i such faces in $\mathcal{A}_i$, each one marked by a point in $P \cap f_i$. The face of $\mathcal{A}_i$ marked by such a point p is a connected component of the intersection of the face marked by p in $\mathcal{A}_i'$ and the face marked by p in $\mathcal{A}_i''$. Since $\mathcal{A}_i'$ can be viewed as part of a line arrangement (as opposed to a line segment arrangement), we can use a result of [3] which implies that the combinatorial complexity of the marked faces in $\mathcal{A}_i'$ is

$$O(m_i^{2/3} n_i^{2/3} + n_i). \tag{2}$$

By Corollary 2.4 the combinatorial complexity of the marked faces in $\mathcal{A}_i''$ is

$$O(m_i k_i^{1/2} \log m_i + k_i \log m_i + k_i \alpha(k_i)). \tag{3}$$

Finally, we can apply the combination lemma, which shows that the combinatorial complexity of the marked faces in $\mathcal{A}_i$ is

$$O(m_i^{2/3} n_i^{2/3} + m_i k_i^{1/2} \log m_i + k_i \log m_i + k_i \alpha(k_i) + m_i + n_i), \tag{4}$$

where the last two terms, m_i and n_i, come from the application of the combination lemma.

Note that by just summing (4) over all funnels f_i we miss some of the edges bounding marked faces in $\mathcal{A}$. To explain how this can happen call a face of $\mathcal{A}_i$ *coastal* if at least one of its edges lies on the boundary of f_i, and call it *inland* otherwise. A marked inland face of $\mathcal{A}_i$ is also a marked face of $\mathcal{A}$. However, a marked coastal face of $\mathcal{A}_i$ is, in general, only a piece of a marked face of $\mathcal{A}$. Fortunately, if a face in $\mathcal{A}$ intersects more than one funnel then it intersects each such funnel in a coastal face. Therefore, we can compensate for any overlooked edges by counting *all* edges of *all* coastal faces of *all* funnels. We obtain a bound on this number by separately considering $\mathcal{A}'_i$ and $\mathcal{A}''_i$, as before. By the "zone theorem" in [6], [7], the combinatorial complexity of all coastal faces in $\mathcal{A}'_i$ is $O(n_i)$. The combinatorial complexity of all coastal faces in $\mathcal{A}''_i$ is $O(k_i\alpha(k_i))$ because by removing the sides of f_i we get all coastal faces as part of one unbounded face in an arrangement of k_i line segments. By the combination lemma the combinatorial complexity of all coastal faces in $\mathcal{A}_i$ is thus $O(n_i+k_i\alpha(k_i))$ which is dominated by (4).

Let us now take the sum of (4) over all funnels f_i. This yields a bound on the combinatorial complexity of all marked faces in $\mathcal{A}$ which is

$$O\left(\sum_{i=1}^{r} m_i^{2/3}n_i^{2/3}+\sum_{i=1}^{r} m_ik_i^{1/2}\log m_i+\sum_{i=1}^{r} k_i\log m_i+\sum_{i=1}^{r} k_i\alpha(k_i)+\sum_{i=1}^{r} m_i+\sum_{i=1}^{r} n_i\right).$$

Using properties (i), (ii) and (iii) we can simplify this expression and obtain

$$O\left(m^{2/3}n^{2/3}+\frac{mn^{1/2}}{r}\log m+n\log m+n\alpha(n)+m+rn\right).$$

We now choose $r=\lceil\frac{m^{1/2}}{n^{1/4}}\log^{1/2}m\rceil$. In this case we have $\frac{mn^{1/2}}{r}\log m\approx rn\approx m^{1/2}n^{3/4}\log^{1/2}m$ and thus

$$K(m,n)=O\left(m^{2/3}n^{2/3}+m^{1/2}n^{3/4}\log^{1/2}m+n\alpha(n)+n\log m\right). \tag{5}$$

Comparing the second term in (5) with the first and the fourth terms, we discover that this bound differs from the asserted bound only if $\sqrt{n}\log n\le m\le\sqrt{n}\log^3 n$. The main reason for falling short of the goal in this interval is the weakness of Corollary 2.4 used to bound the combinatorial complexity of the marked faces in $\mathcal{A}''_i$ resulting in (3). But now we have a new bound so that we can replace (3) by

$$O\left(m_i^{2/3}k_i^{2/3}+m_i^{1/2}k_i^{3/4}\log^{1/2}m_i+k_i\alpha(k_i)+k_i\log m_i\right). \tag{6}$$

After adding (2) and the contribution of the combination lemma we get the following sums expressing the total combinatorial complexity:

$$O\Bigg(\sum_{i=1}^{r} m_i^{2/3}n_i^{2/3}+\sum_{i=1}^{r} m_i^{2/3}k_i^{2/3}+\sum_{i=1}^{r} m_i^{1/2}k_i^{3/4}\log^{1/2}m_i$$
$$+\sum_{i=1}^{r} k_i\alpha(k_i)+\sum_{i=1}^{r} k_i\log m_i+\sum_{i=1}^{r} m_i+\sum_{i=1}^{r} n_i\Bigg).$$

So again we use (i), (ii) and (iii) to simplify this to

$$O\left(m^{2/3}n^{2/3} + \frac{m^{2/3}n^{2/3}}{r^{2/3}} + \frac{m^{1/2}n^{3/4}}{r^{1/2}} \log^{1/2} m + n\alpha(n) + n\log m + m + rn\right).$$

The second and sixth terms can be trivially dropped. If we choose $r = \frac{m^{1/3}}{n^{1/6}} \log^{1/3} m$ we get $\frac{m^{1/2}n^{3/4}}{r^{1/2}} \log^{1/2} m \approx rn \approx m^{1/3}n^{5/6} \log^{1/3} m$. But then, the last term, namely $m^{1/3}n^{5/6}\log^{1/3} m$, is $O(m^{2/3}n^{2/3})$ if $m \geq \sqrt{n}\log n$ and $O(n\log m)$ if $m \leq \sqrt{n}\log^2 n$. We can thus also drop the third and the last terms and get

$$K(m,n) = O\left(m^{2/3}n^{2/3} + n\alpha(n) + n\log m\right),$$

as asserted. ∎

Remarks. (1) It is interesting to observe that the above proof would imply $K(m,n) = O(m^{2/3}n^{2/3} + n\alpha(n))$ if Lemma 2.3 could be improved to $K(m,n) = O(n\alpha(n))$ for $m \leq \sqrt{n}$. It is also interesting that the last step of the current proof, which appears somewhat unnatural, would still be necessary.

(2) In the preceding analysis we assumed that all line segments are bounded, that is, have two endpoints. However, our bounds also apply when some of the line segments are half-lines or lines. Indeed, clip all half-lines and lines outside a sufficiently large circle that encloses all pairwise intersections. This operation replaces all unbounded faces by a single face whose complexity is $O(n\alpha(n))$. Therefore, all bounds above also hold in this slightly more general setting.

3. Sparsely Intersecting Collections of Line Segments

Let S be a set of n line segments in the plane, with a total of t intersecting pairs. In this section we study the dependence on t of the combinatorial complexity of m faces in the arrangement $\mathcal{A}(S)$. We prove an upper bound by combining the techniques of [3] with the results of the preceding section.

Before plunging into the analysis, note that the bound of Theorem 2.5 grossly overestimates the actual combinatorial complexity of m faces when t is considerably smaller than $\binom{n}{2}$. Indeed, if t pairs of line segments intersect, then $\mathcal{A}(S)$ has at most $t+2n$ vertices (the intersection points and the endpoints), at most $2t+n$ edges (each intersection increases the number of edges by two), and, by Euler's formula for planar graphs, at most $t-n+d+1 \leq t+1$ faces, where d is the number of connected components of the union of the n line segments. The total combinatorial complexity of $\mathcal{A}(S)$ is therefore $O(t+n)$. However, if we set $m = \Theta(t)$ in the bound of Theorem 2.5, we obtain $O(t^{2/3}n^{2/3} + n\alpha(n) + n\log t)$, which is much larger than $O(t+n)$ unless t is very small or very large.

We begin with the following observation (Lemma 4.1 of Clarkson and Shor [4]).

Lemma 3.1. *If we draw a random sample R of r line segments from S, the expected number of intersecting pairs in R is $\frac{r(r-1)}{n(n-1)}t$.*

Observe that, if $t < n$, the maximum number of edges of a single face of $\mathcal{A}(S)$ is $\Theta(n)$ since the collection of all faces has combinatorial complexity $\Theta(n)$. In what follows we will therefore assume that $t \geq n$, for otherwise our bounds hold trivially.

Once again, mark the m desired faces of $\mathcal{A}(S)$ by a collection P of m points and fix an integer $1 \leq r \leq n$ to be specified later. Choose a random sample $R \subseteq S$ of r line segments and decompose $\mathcal{A}(R)$ into trapezoidal funnels by drawing a line vertically upward and downward from every vertex (intersection point or endpoint) until it hits another line segment of R or extends to infinity. Let τ denote the number of resulting funnels; τ is clearly proportional to the number of vertices of $\mathcal{A}(R)$, which consist of the $2r$ endpoints of the sample line segments and the intersection points of these line segments. By Lemma 3.1, the expected value of τ is therefore $O(r + \frac{r^2}{n^2}t)$. For each funnel f_i, let m_i be the number of points of P that lie inside f_i, and let n_i be the number of line segments of S that intersect f_i (now we do not make a distinction between line segments that have endpoints in f_i and line segments that do not). Since we are free to perturb the points of P within a sufficiently small neighborhood of their current position, we can assume that no point lies on the boundary of a funnel, and therefore $\sum_{i=1}^{\tau} m_i = m$.

The trapezoidation of $\mathcal{A}(R)$ defines a subdivision of the global problem into smaller subproblems, one for each funnel. As in the proof of Theorem 2.5, for each funnel f_i in the trapezoidation of $\mathcal{A}(R)$, consider the subdivision $\mathcal{A}_i$ of f_i defined by the n_i line segments that intersect f_i. Since no two points of $P \cap f_i$ lie in the same face of A_i we have $m_i \leq \binom{n_i}{2} + n_i + 1$. By the results of the previous section, the combinatorial complexity of the m_i faces of $\mathcal{A}_i$ that contain the m_i points in $P \cap f_i$ is at most

$$O\left(m_i^{2/3} n_i^{2/3} + n_i \alpha(n_i) + n_i \log m_i\right).$$

Recall that a face of $\mathcal{A}_i$ is *coastal* if at least one of its edges lies on the boundary of f_i. As before, we add to this bound the overall complexity of all coastal faces of $\mathcal{A}_i$, which is $O(n_i \alpha(n_i))$. Summing over all funnels, it follows that the combinatorial complexity of the m faces in $\mathcal{A}(S)$ is

$$O\left(\sum_{i=1}^{\tau} m_i^{2/3} n_i^{2/3} + \sum_{i=1}^{\tau} n_i \alpha(n_i) + \sum_{i=1}^{\tau} n_i \log m_i\right). \tag{7}$$

We proceed to bound the expected value of (7). Such an estimate will yield an upper bound on the maximum complexity of m faces of $\mathcal{A}(S)$, as the latter quantity is independent of the choice of the sample.

In the following analysis, we will repeatedly use Theorem 3.6 of Clarkson and Shor [4] which provides a bound on the expected value of expressions of the form $\sum_{i=1}^{\tau} W(\binom{n_i}{d})$, where $W(\cdot)$ is an arbitrary concave non-negative function and d is a positive integer. Interpreted in the context of a random sample of r line segments, the theorem states that, for any fixed d, there is a constant D such that

$$E\left(\sum_{i=1}^{\tau} W(n_i^d)\right) \leq E(\tau) \cdot W\left(D\left(\frac{n}{r}\right)^d\right). \tag{8}$$

By the Hölder inequality, the first term of (7) is at most

$$(9) \qquad O\left(\left(\sum_{i=1}^{\tau} m_i\right)^{2/3}\left(\sum_{i=1}^{\tau} n_i^2\right)^{1/3}\right) = O\left(m^{2/3}\left(\sum_{i=1}^{\tau} n_i^2\right)^{1/3}\right).$$

Now, (8) implies that the expected value of the last sum is

$$E(\tau)\cdot O\left(\left(\frac{n}{r}\right)^2\right) = O\left(\frac{n^2}{r}+t\right),$$

because $E(\tau) = O(r + \frac{r^2}{n^2}t)$ as noted earlier. By Jensen's inequality, for any non-negative random variable X, $E(X^{1/3}) \le (E(X))^{1/3}$, so the expected value of (9) is bounded from above by $O(m^{2/3}(\frac{n^2}{r}+t)^{1/3})$. Moreover, from (8) we also get

$$E(\sum n_i\alpha(n_i)) = E(\tau)\cdot O(\frac{n}{r}\alpha(\frac{n}{r})) = O((n+\frac{r}{n}t)\alpha(\frac{n}{r}))\ ^4,$$

and $E(\sum n_i \log m_i)$ is clearly $O((n+\frac{r}{n}t)\log m)$. On the other hand, recall that $m_i = O(n_i^2)$, for each i, so $E(\sum n_i \log m_i) = O(E(\sum n_i \log n_i))$ which, by another application of (8), is $O(E(\tau)\cdot\frac{n}{r}\log\frac{n}{r}) = O((n+\frac{r}{n}t)\log\frac{n}{r})$. Thus the expected value of the expression in (7) is

$$O\left(m^{2/3}\left(\frac{n^2}{r}+t\right)^{1/3} + \left(n+\frac{r}{n}t\right)\left(\alpha\left(\frac{n}{r}\right)+\min\left\{\log m, \log\frac{n}{r}\right\}\right)\right).$$

If we choose $r = \lceil\frac{n^2}{t}\rceil$ we get $\frac{n^2}{r}+t = O(t)$ and $n+\frac{r}{n}t = O(n)$. It follows that the expected value of the last expression is bounded by

$$O\left(m^{2/3}t^{1/3} + n\alpha\left(\frac{t}{n}\right) + n\min\left\{\log m, \log\frac{t}{n}\right\}\right).$$

Thus we have shown the following upper bound.

Theorem 3.2. *The combinatorial complexity of m faces in the arrangement of a set n line segments with a total of t intersecting pairs is at most*

$$O\left(m^{2/3}t^{1/3} + n\alpha\left(\frac{t}{n}\right) + n\min\left\{\log m, \log\frac{t}{n}\right\}\right).$$

Remarks. (1) The maximum meaningful value of m in Theorem 3.2 is $\Theta(t)$. In this case the above bound becomes $O(t+n)$, which matches the actual combinatorial complexity of the entire arrangement.

(2) When t is quadratic in n, the above bound becomes

$$O\left(m^{2/3}n^{2/3} + n\alpha(n) + n\log m\right),$$

[4] Here it is sufficient to argue that $x\alpha(x) = \sqrt{x^2}\alpha(\sqrt{x^2})$ is a concave function of x^2. Indeed it is not difficult to see that $\sqrt{y}\alpha(\sqrt{y})$ is concave when restricted to the squares of values taken by the Ackermann's function, while our definition of $\alpha(\cdot)$ as a piecewise-linear function guarantees that $\sqrt{y}\alpha(\sqrt{y})$ is concave at intermediate points as well.

which is the same as the bound in Theorem 2.5. When $t \leq n$, on the other hand, our bound reduces to the obvious bound $\Theta(n)$.

(3) When $t \geq n$ and m is about $\frac{n^{3/2}}{t^{1/2}}$, the above bound becomes $O(n \log n)$. Let us compare this with the discussion in [3] concerning "Canham Thresholds", that is, threshold values of m up to which the combinatorial complexity of m faces does not increase much beyond the maximum combinatorial complexity of a single face. We notice that, for the case of line segments, the threshold value of m is roughly $\frac{n^{3/2}}{t^{1/2}}$; it increases from $n^{1/2}$ to n, as the number of intersecting pairs goes down from $\binom{n}{2}$ to n.

Finally, we describe a construction of arrangements defined by sets of $O(n)$ line segments with $O(t)$ intersecting pairs that establish a lower bound on the maximum total number of edges bounding m faces in such arrangements. The bound we obtain is very close to the upper bound demonstrated in Theorem 3.2. The proof of the lower bound is based on constructions given in [12] and in [8].

Theorem 3.3. *For any three positive integers t, m, n such that $m \leq t-n+2$ and $t \leq \binom{n}{2}$, there exists an arrangement of $O(n)$ line segments with $O(t)$ intersecting pairs in which there are m or fewer faces with combinatorial complexity $\Omega(m^{2/3}t^{1/3}+n\alpha(\frac{t}{n}))$.*

Proof. We first take care of the case $t < n$. As mentioned before, the entire arrangement has $\Theta(n)$ edges in this case. In fact, $t < n$ is small enough so that there exists an arrangement of n line segments with t intersecting pairs in which there are no bounded faces. Hence in such an arrangement, there are m or fewer faces (namely one face) of complexity $\Theta(n)$, as claimed. For the remainder of the proof we can thus assume $t \geq n$.

By the construction of [12], for any integer k, there exists a set S_k of k line segments so that the unbounded face of the arrangement $\mathcal{A}(S_k)$ has $\Omega(k\alpha(k))$ edges. Take $q = \lceil \frac{n^2}{t} \rceil$ disjoint translates of S_k, with $k = \lceil \frac{n}{q} \rceil$, so that no line segment of one translate intersects any line segment of any other translate. This gives an arrangement defined by $O(n)$ line segments in which the unbounded face has $\Omega(q \cdot \frac{n}{q}\alpha(\frac{n}{q})) = \Omega(n\alpha(\frac{t}{n}))$ edges. Moreover, the number of intersecting pairs in any copy of S_k is at most $\binom{k}{2} = O(\frac{n^2}{q^2}) = O(\frac{t^2}{n^2}) = O(t/q)$, so that the total number of intersecting pairs is $O(t)$, as desired. This yields the second term of the claimed lower bound.

We can thus assume that $m > \frac{n^{3/2}}{t^{1/2}}$, for otherwise the first term of our bound is dominated by the second one. By the construction of [8], for every pair of positive integers, (k,ℓ), such that $k^2 \geq \ell$, there exists a set T_k of k line segments so that some ℓ faces in $\mathcal{A}(T_k)$ have combinatorial complexity $\Omega(k^{2/3}\ell^{2/3})$. Apply this construction with $k = \lceil \frac{n}{q} \rceil$ and $\ell = \lfloor \frac{m}{q} \rfloor$, with q as above, and consider q disjoint translates of T_k. The resulting set contains $\Theta(n)$ line segments and the number of intersecting pairs is at most $q\binom{k}{2} = O(t)$. The defined arrangement has $q\ell \leq m$ faces with combinatorial complexity

$$\Omega\left(qk^{2/3}\ell^{2/3}\right) = \Omega\left(\frac{n^2}{t}\left(\frac{t}{n}\right)^{2/3}\left(\frac{mt}{n^2}\right)^{2/3}\right) = \Omega\left(m^{2/3}t^{1/3}\right).$$

These calculations are valid because $q \leq n$ and $q \leq m$ for the assumed range of t, m, and n, so $k \geq 1$ and $\ell \geq 1$. ∎

4. Conclusions

This paper proves an upper bound on the total number of edges bounding m faces in an arrangement of n line segments in the plane, which improves the previous upper bound of [5]. In addition, we show how the maximum number of edges depends on the number of intersecting pairs of the given line segments. Our analysis makes a more sophisticated use of the combination lemma of [5], but we do not know whether the results that we obtain are best possible using this technique. Our bounds come close to the known lower bounds (or to those established in Theorem 3.3) but leave a gap for some range of values of m.

An obvious open problem is to close this gap. It appears that progress in this direction hinges on a better understanding of $K(m,n)$ for $m = n^{1/2}$. Using the techniques of [3], an improved bound on $K(n^{1/2},n)$ would immediately yield better bounds for the general case. In view of the apparent difficulty in obtaining such a bound, a plausible conjecture is that $K(n^{1/2},n)$ is indeed asymptotically larger than the known lower bound which is $\Omega(n\alpha(n))$. If so, that would represent yet another surprising difference between arrangements of line segments and arrangements of lines; in an arrangement of n lines $n^{1/2}$ faces cannot have asymptotically more edges than a single face.

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